Modulational instability in asymmetric coupled wave functions

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Abstract. The evolution of the amplitude of two nonlinearly interacting waves is considered, via a set of coupled nonlinear Schrödinger-type equations. The dynamical profile is determined by the wave dispersion laws (i.e. the group velocities and the group velocity dispersion terms) and the nonlinearity and coupling coefficients, on which no assumption is made. A generalized dispersion relation is obtained, relating the frequency and wave-number of a small perturbation around a coupled monochromatic (Stokes') wave solution. Explicitly stability criteria are obtained. The analysis reveals a number of possibilities. Two (individually) stable systems may be destabilized due to coupling. Unstable systems may, when coupled, present an enhanced instability growth rate, for an extended wave number range of values. Distinct unstable wavenumber windows may arise simultaneously.

Amplitude modulation (AM) is a widely known nonlinear mechanism dominating wave propagation in dispersive media [1]; it is related to mechanisms such as modulational instability (MI), harmonic generation and energy localization, and possibly leads to soliton formation. The study of AM generically relies on nonlinear Schrödinger (NLS) type equations [2]; a set of coupled NLS (CNLS) equations naturally occurs when interacting modulated waves are considered. CNLS equations are encountered in physical contexts as diverse as electromagnetic wave propagation [3,4], optical fibers [5,6], plasma waves [7–9], transmission lines [10], and left-handed (negative refraction index) metamaterials [11]. A similar mathematical model is employed in the mean-field statistical mechanical description of boson gases, to model the dynamics of Bose-Einstein condensates [12–14]. In this paper, we shall investigate the (conditions for the) occurrence of MI in a pair of (asymmetric) CNLS equations, from first principles. A set of stability criteria are derived, to be tailor cut to a (any) particular problem of coupled wave propagation.

1 The model

Let us consider two coupled waves propagating in a dispersive nonlinear medium. The wave functions ($j = 1, 2$)

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are modelled by $\psi_j \exp i(\mathbf{k_j}\mathbf{r} - \omega_j t) + $ c.c. (complex conjugate), where the carrier wave number $\mathbf{k_j}$ and frequency ω_j of each wave are related by a dispersion relation function $\omega_j = \omega_j(\mathbf{k_j})$. Nonlinearity is manifested via a slow modulation of the wave amplitudes, in time and space, say along the x-axis. The amplitude evolution is described by a pair of CNLS Eqs.

$$i\left(\frac{\partial\psi_1}{\partial t} + v_{g,1}\frac{\partial\psi_1}{\partial x}\right) + P_1\frac{\partial^2\psi_1}{\partial x^2}$$
$$+ Q_{11}|\psi_1|^2\psi_1 + Q_{12}|\psi_2|^2\psi_1 = 0,$$
$$i\left(\frac{\partial\psi_2}{\partial t} + v_{g,2}\frac{\partial\psi_2}{\partial x}\right) + P_2\frac{\partial^2\psi_2}{\partial x^2}$$
$$+ Q_{22}|\psi_2|^2\psi_2 + Q_{21}|\psi_1|^2\psi_2 = 0. \quad (1)$$

The group velocity $v_{g,j}$ and the group-velocity-dispersion (GVD) term P_j corresponding to the j−th wave is related to (the slope and the curvature, respectively, of) the dispersion curve via $v_{g,j} = \partial\omega_j/\partial k_{j,x}$ and $P_j = \partial^2\omega_j/2\partial k_{j,x}^2$ (differentiation *in the direction of modulation*). Q_{jj} and $Q_{jj'}$ model carrier self-modulation and wave coupling, respectively. No hypothesis holds, a priori, on the sign and/or the magnitude of either of these coefficients. The group velocities are often assumed equal, in which case (and only) the corresponding terms are readily eliminated via a Galilean transformation. The combined assumption $P_1 = P_2$, $Q_{11} = Q_{22}$ and $Q_{12} = Q_{21}$ is often made in nonlinear optics [4,15]. The case $P_1 = P_2$,

$Q_{11} = Q_{21}$ and $Q_{12} = Q_{22}$ was also recently reported, in negative refarction index composite metamaterials [11].

2 Modulational (in)stability of single waves

Let us first briefly outline, for later reference, the analysis in the case of a single modulated wave, say here recovered by setting $\psi_2 = 0$ in equations (1); the (single) NLS equation is thus obtained. According to the standard formalism [1,2], ψ_1 ($= \psi$, dropping the index in this paragraph) is *modulationally unstable* (stable) if $PQ > 0$ ($PQ < 0$). To see this, one may first check that the NLS equation is satisfied by the plane wave solution $\psi(x,t) = \psi_0 \, e^{iQ|\psi_0|^2 t}$, where ψ_0 is a constant (real) amplitude. The standard (linear) stability analysis then shows that a linear perturbation (say $\psi_0 \to \psi_0 + \delta\psi$, with $\delta\psi = \epsilon\psi_1$, where $\epsilon \ll 1$) with frequency Ω and wavenumber K [i.e. $\psi_1 \sim \exp i(Kx - \Omega t)$] obeys the dispersion relation: $(\Omega - v_g K)^2 = P K^2 \left(P K^2 - 2Q|\psi_0|^2 \right)$, which exhibits a purely growing unstable mode if $K \leq K_{cr,0} = (2Q/P)^{1/2}|\psi_0|$ (hence only if $PQ > 0$). The growth rate $\sigma = \mathrm{Im}\,\Omega$ attains a maximum value $\sigma_{max} = Q|\psi_0|^2$ at $K_{cr,0}/\sqrt{2}$. For $PQ < 0$, on the other hand, the wave is *stable* to external perturbations.

3 Coupled wave stability analysis

In order to investigate the modulational stability profile of a pair of coupled waves, we shall first seek an equilibrium state in the form $\psi_j = \psi_{j0} \exp[i\varphi_j(t)]$ (for $j = 1, 2$), where ψ_{j0} is a (constant real) amplitude and $\varphi_j(t)$ is a (real) phase, into equations (1). We thus find a monochromatic (fixed-frequency) solution of the form $\varphi_j(t) = \Omega_{j0}t$, where $\Omega_{j0} = Q_{jj}\psi_{j0}^2 + Q_{jl}\psi_{l0}^2$ (for $j \neq l = 1, 2$, henceforth understood everywhere). Considering a small perturbation around equilibrium, we take $\psi_j = (\psi_{j0} + \delta\psi_j) \exp[i\varphi_j(t)]$, where $\delta\psi_j = \epsilon\psi_{j1}(\mathbf{r}, t)$ is a small ($\epsilon \ll 1$) complex amplitude perturbation of the wave amplitudes. Substituting into equations (1) and separating real and imaginary parts by writing $\psi_{j1} = a_j + ib_j$ (where $a_j, b_j \in \Re$), the first order terms (in ϵ) yield

$$-\frac{\partial b_j}{\partial t} - v_{g,j}\frac{\partial b_j}{\partial x} + P_j\frac{\partial^2 a_j}{\partial x^2} + 2Q_{jj}\psi_{j0}^2 a_j$$
$$+2Q_{jl}\psi_{j0}\psi_{l0}a_l = 0,$$
$$\frac{\partial a_j}{\partial t} + v_{g,j}\frac{\partial a_j}{\partial x} + P_j\frac{\partial^2 b_j}{\partial x^2} = 0. \quad (2)$$

Eliminating b_j, these equations yield

$$\left[\left(\frac{\partial}{\partial t} + v_{g,j}\frac{\partial}{\partial x}\right)^2 + P_1\left(P_1\frac{\partial^2}{\partial x^2} + 2Q_{11}\psi_{10}^2\right)\frac{\partial^2}{\partial x^2}\right]a_1$$
$$+ 2P_1 Q_{12}\psi_{10}\psi_{20}\frac{\partial^2}{\partial x^2}a_2 = 0, \quad (3)$$

(along with a symmetric equation, upon $1 \leftrightarrow 2$). Let $a_j = a_{j0}\exp[i(Kx - \Omega t)]$+ c.c., where K and Ω are

the wavevector and the frequency of the perturbation, respectively, viz. $\partial/\partial t \to -i\Omega$ and $\partial/\partial x \to iK$. We thus obtain an eigenvalue problem in the form $\mathbf{Ma} = \Omega^2 \mathbf{a}$, where $\mathbf{a} = (a_{10}, a_{20})^T$ and the matrix elements are $M_{jj} = P_j K^2 (P_j K^2 - 2Q_{jj}\psi_{j0}^2) \equiv \Omega_j^2$ and $M_{jl} = -2P_j Q_{jl}\psi_{j0}\psi_{l0}K^2$ (for $l \neq j = 1$ or 2)

$$\left[(\Omega - v_{g,1}K)^2 - \Omega_1^2\right]\left[(\Omega - v_{g,2}K)^2 - \Omega_2^2\right] = \Omega_c^4 \quad (4)$$

where $\Omega_c^4 = M_{12}M_{21}$. This dispersion relation is a 4th order polynomial equation in Ω.

4 Equal group velocities

For $v_{g,1} = v_{g,2}$, setting $\Omega - v_{g,1/2}K \to \Omega$ reduces (4) to

$$\Omega^4 - T\Omega^2 + D = 0, \quad (5)$$

where $T = \mathrm{Tr}\mathbf{M} \equiv \Omega_1^2 + \Omega_2^2$ and $D = \mathrm{Det}\mathbf{M} \equiv \Omega_1^2\Omega_2^2 - \Omega_c^4$ are the *trace* and the *determinant*, respectively, of the matrix $\mathbf{M}$. Equation (5) admits the solution $\Omega^2 = \frac{1}{2}[T \pm (T^2 - 4D)^{1/2}]$, or

$$\Omega_\pm^2 = \frac{1}{2}(\Omega_1^2 + \Omega_2^2) \pm \frac{1}{2}\left[(\Omega_1^2 - \Omega_2^2)^2 + 4\Omega_c^4\right]^{1/2}. \quad (6)$$

Stability is ensured (for any wavenumber K) if (and only if) *both* of the (two) solutions of (5), say $\Omega_\pm^2$, are *positive* (real) numbers. In order for the right-hand side to be real, the *discriminant* quantity $\Delta = T^2 - 4D = (\Omega_1^2 - \Omega_2^2)^2 + 4\Omega_c^4$ has to be positive. Furthermore, recalling that the roots of the polynomial $p(x) = x^2 - Tx + D$, say $r = r_{1,2}$, satisfy $T = r_1 + r_2$ and $D = r_1 r_2$, the stability requirement is tantamount to the following three conditions being satisfied simultaneously: $T > 0$, $D > 0$ and $\Delta = T^2 - 4D > 0$.

The first stability condition, namely the positivity of the trace T: $T = K^2[K^2 \sum_j P_j^2 - 2\sum_j P_j Q_{jj}\psi_{j0}^2] > 0$, depends on (the sign of) the quantity $q_1 \equiv \sum_j P_j Q_{jj}|\psi_{j0}|^2$ which has to be negative for stability. The only case ensuring *absolute* stability (for any ψ_{j0} and k) is

$$P_1 Q_{11} < 0 \quad \text{and} \quad P_2 Q_{22} < 0. \quad (7)$$

Otherwise, T becomes negative (and thus either $\Omega_-^2 < 0 < \Omega_+^2$ or $\Omega_-^2 < \Omega_+^2 < 0$) for K below a critical value $K_{cr,1} = (2\sum_j P_j Q_{jj}\psi_{j0}^2/\sum_j P_j^2)^{1/2} > 0$ (cf. the single wave criterion above); this is always possible for a sufficiently large perturbation amplitude $|\psi_{20}|$ if, say, $P_2 Q_{22} > 0$ (even if $P_1 Q_{11} < 0$). Therefore, only a pair of two individually stable waves can be stable, or the presence of a *single* unstable wave may de-stabilize its counterpart.

The second stability condition, namely positivity of the determinant D, amounts to

$$D(K^2) = P_1 P_2 K^4 \big[(P_1 K^2 - 2Q_{11}\psi_{10}^2)(P_2 K^2 - 2Q_{22}\psi_{20}^2)$$
$$- 4Q_{12}Q_{21}\psi_{10}^2\psi_{20}^2\big] > 0.$$

We see that $D(K^2)$ bears two non-zero roots for K^2, namely $K_{D,1/D,2}^2 = \frac{1}{2P_1^2 P_2^2}[q_2 \mp (q_2^2 - 4P_1^2 P_2^2 q_3)^{1/2}]$, as obvious from the expanded form $D = k^4(P_1 P_2 K^4 - q_2 K^2 + q_3)$, where $q_2 \equiv \sum K_{D,j}^2 = 2P_1 P_2(P_2 Q_{11}\psi_{10}^2 + P_1 Q_{22}\psi_{20}^2)$ and $q_3 \equiv \prod K_{D,j}^2 = 4P_1 P_2(Q_{11}Q_{22} - Q_{12}Q_{21})\psi_{10}^2\psi_{20}^2$. The condition $D > 0$ ($\forall K \in \Re$) requires either:

– that the discriminant quantity $\Delta' = q_2^2 - 4P_1^2 P_2^2 q_3 = 4P_1^2 P_2^2[(P_1 Q_{22}\psi_{20}^2 - P_2 Q_{11}\psi_{10}^2)^2 + 4P_1 P_2 Q_{12}Q_{21}\psi_{10}^2\psi_{20}^2]$ be non-positive, i.e. $\Delta' \le 0$ (this is only possible if $P_1 P_2 Q_{12}Q_{21} < 0$ *and* for a specific relation to be satisfied by the perturbation amplitudes ψ_{j0}; that is, it *cannot* be generally satisfied, $\forall \psi_{j0}$), or

– that $\Delta' > 0$ and *both* of the (real) non-zero roots $K_{D,1/D,2}^2$ of $D(K^2)$ be *negative*; this is ensured if $q_2 < 0$ *and* $q_3 > 0$. If $q_3 > 0$ and $q_2 > 0$, then the two roots $K_{D,1/D,2}^2$ will be positive ($0 < K_{D,1}^2 < K_{D,2}^2$) and the wave pair will be unstable to a perturbation with intermediate K, i.e. $K_{D,1}^2 < K^2 < K_{D,2}^2$. If $q_3 < 0$ (regardless of q_2), then $K_{D,1}^2 < 0 < K_{D,2}^2$, and the wave pair is unstable to a perturbation with $K^2 < K_{D,2}^2$.

These instability scenaria and wavenumber thresholds are sufficient for symmetric wave systems (i.e. upon $1 \leftrightarrow 2$), as we shall see below.

The last stability condition regards the positivity of the discriminant quantity $\Delta = T^2 - 4D$ (irrelevant if $D < 0$). We consider the inequality

$$\Delta(K^2) = K^4 (d_4 K^4 - d_2 K^2 + d_0) > 0$$

where $d_4 = (P_1^2 - P_2^2)^2$, $d_2 = 4(P_1^2 - P_2^2)(P_1 Q_{11}\psi_{10}^2 - P_2 Q_{22}\psi_{20}^2)$ and $d_0 = 4[(P_1 Q_{11}\psi_{10}^2 - P_2 Q_{22}\psi_{20}^2)^2 + 4P_1 P_2 Q_{12}Q_{21}\psi_{10}^2\psi_{20}^2]$. We should distinguish two cases here.

If $P_1 = P_2 = P$, this condition reduces to $d_0 > 0$, i.e. here $d_0 = 4P^2[(Q_{11}\psi_{10}^2 - Q_{22}\psi_{20}^2)^2 + 4Q_{12}Q_{21}\psi_{10}^2\psi_{20}^2] \equiv 4P^2 q_4$. Stability (for all ψ_{j0}) is thus only ensured if $Q_{12}Q_{21} > 0$. For symmetric wave pairs, i.e. for $P_1 = P_2 = P$ and $Q_{12} = Q_{21}$, this last necessary condition for stability is always fulfilled. If, on the other hand, $Q_{12}Q_{21} \le 0$, the wave pair will be unstable in a range of values (e.g. of the ratio ψ_{10}/ψ_{20}), to be determined by solving $d_0 < 0$).

Let us now assume (with no loss of generality) that $P_1 > P_2$. Since $\Delta'' = d_2^2 - 4d_4 d_0 = -64P_1 P_2 Q_{12}Q_{21}(P_1^2 - P_2^2)^2\psi_{10}^2\psi_{20}^2$, the stability condition $\Delta > 0$ is satisfied *for all K and ψ_{j0}* only if the quantity $q_5 \equiv P_1 P_2 Q_{12}Q_{21}$ is positive, hence $\Delta'' < 0$; again, this is always true for symmetric waves. Now if, on the other hand, $q_5 < 0$ (i.e. $\Delta'' > 0$), then one needs to investigate the sreduceigns of $d_2 = K_{\Delta,1}^2 + K_{\Delta,2}^2 \equiv q_6$ and $d_0 = K_{\Delta,1}^2 K_{\Delta,2}^2 \equiv q_7$, in terms of the amplitudes ψ_{j0}. Here, $K_{\Delta,1/\Delta,2}^2 = \frac{1}{2d_4}[d_2 \mp \sqrt{d_2^2 - 4d_4 d_0}]$. Similar to the analysis of the previous condition (see above), one may easily see that both signs are possible for both quantities d_2 and d_0. The only possibility for stability ($\forall K$) is provided by the combination $d_2 < 0$ and $d_0 > 0$ (hence $K_{\Delta,1}^2 < K_{\Delta,2}^2 < 0$). The possibility for instability arises either for $K_{\Delta,1}^2 < 0 < K^2 < K_{\Delta,2}^2$
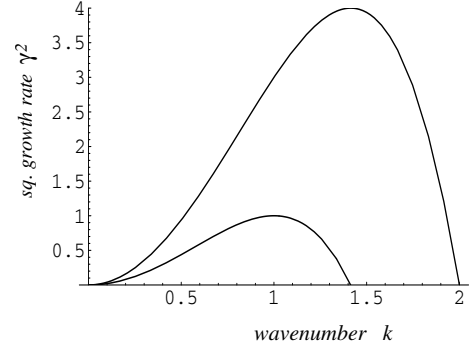

Fig. 1. The square of the instability growth rate $\gamma^2 \equiv -\Omega_-^2$ is depicted versus the perturbation wavenumber K (arbitrary parameter values). Notice the difference from the single wave case (lower curve).

(if $d_0 < 0$), or for $0 < K_{\Delta,1}^2 < K^2 < K_{\Delta,2}^2$ (if $d_0 > 0$ *and* $d_2 > 0$). As above, see that we obtain the possibility for a window of instability far from $K = 0$.

Instability is manifested as a purely growing mode, when one or more of the above conditions are violated. In specific, if $T < 0$ and/or $D < 0$, then one (or both) of the solutions of the dispersion relation (4) (for Ω^2) becomes negative, say $\Omega_-^2 < 0$ [given by (6)]; the instability growth rate in this case is given by $\sigma \equiv \sqrt{-\Omega_-^2}$, and is manifested in the wavenumber ranges $[0, K_{cr,1}]$ and either $[0, K_{D,2}]$ or $[K_{D,1}, K_{D,2}]$ (depending on parameter values; see the definitions above).

If $\Delta = T^2 - 4D < 0$, on the other hand (hence $D > 0$), then all solutions of (4) are complex, thus developing an imaginary part $\text{Im}(\Omega_\pm^2) = \pm\sqrt{|\Delta|}/2$, so (the maximum value of) $\text{Im}(\Omega_\pm) = \text{Im}(\Omega_\pm^2)^{1/2}$ gives the instability growth rate σ. As found above, this will be possible for wave numbers either in $[0, K_{\Delta,2}]$ or $[K_{\Delta,1}, K_{\Delta,2}]$ (see the definitions above).

The analysis indicates that up to three different unstable wavenumber "windows" may appear; these windows may either be partially superposed, or distinct from each other. One may therefore qualitatively anticipate MI occurring for $K \in [0, K_{cr}]$ (some threshold) and, *also*, for $K \in [K'_{cr}, K''_{cr}]$ (K'_{cr} may be higher or lower than K_{cr}, depending on the problem's parameters). Furthermore, the instability growth rate witnessed may be dramatically modified by the coupling, both quantitatively (higher rate) and qualitatively (enlarged unstable wavenumber region); see in Figure 1.

Summarizing the above results, should one wish to investigate the occurrence of modulational instability in a given physical problem, one has to verify condition (7), and then consider the (sign of the) quantities $q_1, ...q_7$ (defined above).

5 The role of the group velocity misfit

It may be interesting to discuss the role of the group velocity difference, in a coupled wave system. Keeping the
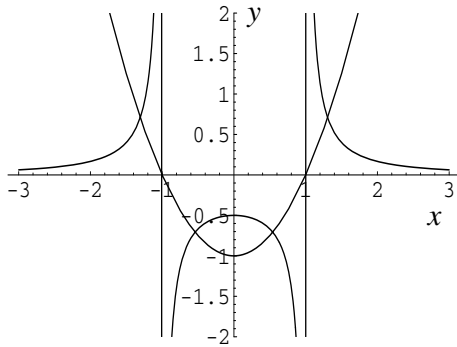
Fig. 2. The functions $f_1(x)$ (parabola) and $f_2(x)$ (rational function, two vertical asymptotes) defined in the text are depicted, vs. x, for $A = B = -1$, $C = 0.5$ (so that $D = AB - C = +0.5 > 0$), $x_1 = x_2 = 0$ (equal group velocities). Note that a group velocity mismatch (a horizontal shift) may destabilize a pair of (stable, separately) waves (i.e. reduce the intersection points from 4 to 2).

Fig. 3. The functions $f_1(x)$ and $f_2(x)$ are depicted, for $A = B = +1$, $C = 0.5$ (so that $D = AB - C > 0$), and $x_1 = x_2 = 0$. At most 2 intersection points may occur by translation. A pair of (unstable, separately) waves is always unstable.

discussion qualitative, we shall avoid to burden the presentation with tedious numerical calculations. One may rather point out the role of the group velocity misfit via simple geometric arguments. Inspired by an idea proposed in reference [7], we may express the general dispersion relation (4) in the form

$$f_1(x) = f_2(x) \tag{8}$$

where we have defined the functions $f_1(x) = (x - x_1)^2 + A$ and $f_2(x) = \frac{C}{(x - x_2)^2 + B}$, and the real quantities $x_j = K v_{g,j}$, $A = -\Omega_1^2 = -M_{11}$, $B = -\Omega_2^2 = -M_{22}$, and $C = \Omega_c^4 = M_{12} M_{21}$; x here denotes Ω. The stability profile is determined by the number of *real* solutions of equation (8), an integer, say r, between 0 and 4. For absolute stability (for any K, $|\psi_{j0}|$), we need to have 4 real solutions; in any other case, i.e. if $r < 4$, the (imaginary part of) the $4 - r$ complex solutions determine(s) the growth rate of the instability. Note that $x_1 \neq x_2$ expresses the group velocity mismatch $v_{g,1} \neq v_{g,2}$. Negative A (B) means that wave 1 (2) alone is stable, and vice versa.

Let us first consider a wave pair satisfying $C > 0$, i.e. for $M_{12} M_{21} \sim P_1 P_2 Q_{12} Q_{21} > 0$ (a *symmetric* wave pair, for instance). We shall study the curves representing the functions $f_1(x)$ and $f_2(x)$ on the xy plane. The former one is a parabola, with a minimum at (x_1, A). The latter one is characterized by a local maximum (for $C > 0$) at $(x_2, C/B)$, in addition to a horizontal asymptote (the x-axis), since $f_2(x) \to 0$ for $x \to \pm\infty$. Furthermore, for $B < 0$ (only), $f_2(x)$ has two vertical asymptotes (poles) at $x = x_2 \pm \sqrt{|B|}$ (see Figs. 2, 3). Now, for a *stable - stable* wave pair (i.e. for $A, B < 0$), we have seen that the dispersion relation (4) predicted stability. This result regarded the *equal* (or vanishing) group velocity case, $v_{g,1} = v_{g,2}$, and may be visualized by plotting $f_1(x)$ and $f_2(x)$ for $x_1 = x_2$ and $A, B < 0$; see Figure 2a. Let us first assume that $D = M_{11} M_{22} - M_{12} M_{21} = AB - C$ is *positive*, implying (for $B < 0$) that $A < C/B$; thus, the minimum of $f_1(x)$ lies below the local maximum of $f_2(x)$. Thus, 4 points of intersection exist (cf. Fig. 2), for $x_1 = x_2$ and $A, B < 0$;
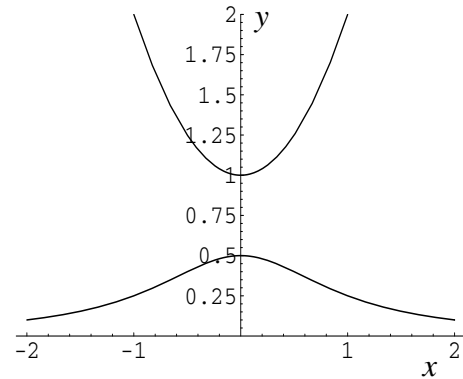
this fact ensures stability, as we saw above via analytical arguments, for $v_{g,1} = v_{g,2}$ and $M_{11}, M_{22} > 0$ (both waves individually stable). Now, considering $v_{g,1} \neq v_{g,2}$ results in a horizontal shift between the two curves (cf. Fig. 2), which may exactly result in reducing the number of intersection points from 4 to 2 (enabling instability). Therefore, *a pair of stable waves may be destabilized due to a finite difference in group velocity.*

Still for a stable-stable wave pair ($A, B < 0$), let us assume that $D = AB - C < 0$, implying (for $B < 0$) that $A > C/B$. Thus, the minimum of $f_1(x)$ here lies *above* the local maximum of $f_2(x)$, and only 2 points of intersection now exist, (shift the parabola upwards in Fig. 2 to see this); this fact imposes *instability* (for $D < 0$), as predicted above.

Considering an *unstable - unstable* wave pair (i.e. $A > 0$ and $B > 0$) with $D = AB - C > 0$ ($A > C/B$). Plotting $f_1(x)$ and $f_2(x)$ for $x_1 = x_2$ and $A, B > 0$ (see Fig. 3), we see that the minimum of $f_1(x)$ lies above the local maximum of $f_2(x)$. *No* points of intersection exist, a fact which prescribes *instability*. Considering $v_{g,1} \neq v_{g,2}$ simply results in a horizontal shift between the two curves, which *does not* affect this result at all. On the other hand (still for $A > 0$ and $B > 0$), now assuming that $D = AB - C < 0$, i.e. $A < C/B$, results in a vertical shift downwards of the parabola in Figure 3; at least 2 complex solutions obviously exist, hence instability. Therefore, *a pair of unstable waves is always unstable* ($\forall v_{g,1}, v_{g,2}$).

Still for $C > 0$, one may consider a *stable - unstable* wave pair (say, for $A < 0$ and $B > 0$, with no loss of generality): the plot of f_1 and f_2 (here omitted) would look like Figrue 3 upon a strong vertical translation of the parabola downwards (so that the minimum lies in the lower half-plane, since $A < 0$). Instability ($r = 2$) dominates this case also.

Let us now consider a wave pair satisfying $C < 0$, i.e. $M_{12} M_{21} \sim P_1 P_2 Q_{12} Q_{21} = P^2 Q^2 < 0$; this has to be an *asymmetric* wave pair. Again, different cases may be distinguished.

For an *stable-unstable* wave pair, i.e. say for $A < 0 < B$, different possibilities exist: cf. Figure 4, where 4 points
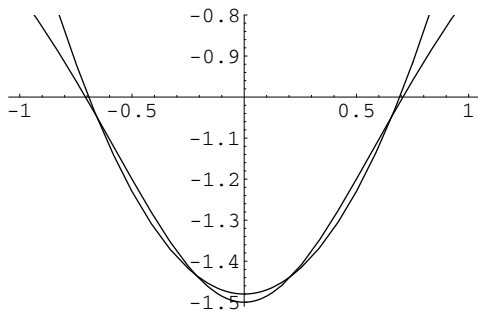
Fig. 4. *Stable-unstable* wave interaction: the functions $f_1(x)$ and $f_2(x)$ are depicted, for $A = -1.48$, $B = +1$, $C = -1.5$, and $x_1 = x_2 = 0$. This (stable, 4 intersection points) configuration may be destabilized either by a horizontal (v_g difference) or a vertical (A value) shift.
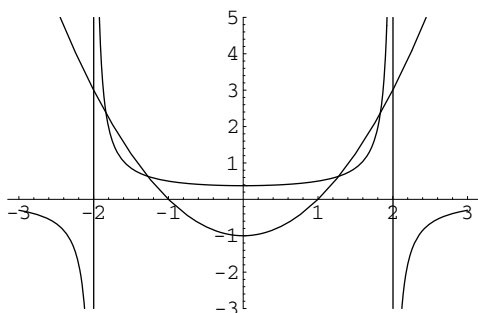

Fig. 5. *Stable-stable* wave interaction: the functions $f_1(x)$ and $f_2(x)$ are depicted, for $A = -1$, $B = -4$, $C = -1.5$ and $x_1 = x_2 = 0$. This (stable, 4 intersection points) configuration may be destabilized either by a horizontal (v_g difference) or a vertical (A value) shift.

of intersection ensure stability, for $v_{g,1} \approx v_{g,2}$). However, either a (horizontal) shift v_g difference or (a vertical shift) in A may render the system unstable.

For $A, B > 0$ (both waves intrinsically *unstable*), one easily sees that no intersection occurs (figure omitted; simply translate the parabola upwards in Fig. 4); the pair is unstable.

Finally, for a *stable-stable* wave pair ($A, B < 0$), the wave pair may stable (see Fig. 5); this configuration is nevertheless destabilized either by a velocity misfit (a horizontal shift) or a vertical shift (in A).

In conclusion, we have investigated the occurrence of modulational instability in a pair of coupled waves, co-propagating and interacting with one another. Relying on a coupled nonlinear Schrödinger equation model, we have derived a complete set of explicit (in)stability criteria, in addition to exact expressions for the critical wavenumber thresholds. Furthermore, we have traced the role of the group velocity mismatch on the coupled wave stability. The results are readily applied to a set of coupled Gross-Pitaevskii equations (modelling a pair of BECs in condensed boson gases), as exposed here, as well as in a variety of physical situations.

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